Int. J. Solids Structures, 1970, Vol. 6, pp. 1535 to 1543. Pergamon Press. Printed in Great Britain

ELASTIC INCLUSION PROBLEMS IN PLANE ELASTOSTATICS†

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Abstract—A general representation of the solution of the elastic curvilinear inclusion problem is given. It is shown that an ellipse and its geometric limits is the only inclusion shape for which a uniform stress applied at infinity induces a constant state of stress within the inclusion. The application of the general representation of the solution is illustrated by a number of examples.

INTRODUCTION

PROBLEMS of determining the stress distributions induced by inclusions and cavities in loaded plane members have aroused considerable interest for over half a century. Myriads of works have appeared dealing with these problems as may be seen from the excellent bibliographies given in the books by Muskhelishvili [1] and Savin [2].

Even though the problem of finding stress distributions around curvilinear cavities has been treated systematically [2], the corresponding problems for rigid inclusions have received relatively little attention [1–7]. If the inclusion is elastic,‡ the situation is even worse with analytical results limited to cases of elliptic and circular geometries.

The case of an elastic elliptic inclusion in an isotropic matrix seems to have been originally treated by Donnell [11], who used elliptic coordinates. The analysis was extended to the thermoelastic problem by Mindlin and Cooper [12]. More recently, Hardiman [13] used the complex variables method to treat the elliptic inclusion problem and, apparently, he was the first to notice that a uniform applied load at infinity induces a constant state of stress within an elliptic inclusion. Subsequently, the problem was solved, using an energy based method, by Bhargava and Radhakrishna [14, 15]. Stippes [16] has proved that the elastic elliptic inclusion and its geometric limits is the only inclusion shape for which a homogeneous stress field, applied at infinity, induces a constant state of stress within the inclusion. The situation in the case of a circular inclusion is considerably better, with problems as complicated as an edge dislocation near and inside such inclusions having been treated successfully [17].

In the present work, the *elastic curvilinear inclusion problem is considered and the general form of the solution is established*. This solution exhibits the dependence on the elastic constants of the matrix and inclusion that was recently established by Dundurs [18]. The

† The work leading to this paper was supported by the Advanced Research Projects Agency, Department of Defense through Grant No. AF 33(615)-3110, administered by the Air Force Materials Laboratory.

‡ It should be noted that a number of papers have appeared recently supposedly dealing with the curvilinear inclusion problem (see for example, [8, 9]). In these, the term "inclusion" is used in the sense of Eshelby [10]; that is, an inclusion is a region of a homogeneous elastic solid that has undergone inelastic or stress free deformation.

general results are specialized for particular combinations of materials, namely, a curvilinear cavity or inclusion of vanishing rigidity and a rigid curvilinear inclusion. Furthermore, it is shown that an ellipse and its geometric limits is the only inclusion shape for which a uniform stress applied at infinity induces a constant state of stress within the inclusion follows directly from the general solution. The application of the general results is illustrated by a number of examples.

GENERAL RESULTS

Consider an isotropic solid in a state of plane deformation parallel to the x, y plane. The displacements and stresses in the plane of deformation can be written in terms of two functions of the complex variable $z = x+iy$ as

$$\left.\begin{aligned} 2\mu(u+iv) &= \kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}, \\ \sigma_{xx}-i\sigma_{xy} &= \phi'(z)+\overline{\phi'(z)}-\bar{z}\phi''(z)-\psi'(z), \\ \sigma_{xx}+\sigma_{yy} &= 2[\phi'(z)+\overline{\phi'(z)}], \end{aligned}\right\} \tag{1}$$

where a prime on a function is used to denote differentiation with respect to its argument and

$$\kappa = \begin{cases} 3-4\nu & \text{for plane strain,} \\ (3-\nu)/(1+\nu) & \text{for plane stress;} \end{cases} \tag{2}$$

μ and ν are the shear modulus and Poisson's ratio, respectively [1].

Curvilinear coordinates

Let z and ζ be two complex variables related by the transformation

$$z = \omega(\zeta) = c\left[\zeta+\sum_{n=1}^{\infty}\lambda_n\zeta^{-n}\right]. \tag{3}$$

Equation (3) maps the unit circle $|\zeta| = 1$, or $\rho = 1$ where $\zeta = \rho e^{i\theta}$, in the ζ-plane into a closed curvilinear contour Σ in the z-plane. *Points exterior to the circle $|\zeta| = 1$ are mapped uniquely into points exterior to Σ if and only if the coefficients λ_n are restricted appropriately.* Determination of the necessary restrictions on the coefficients λ_n is beyond the scope of this work; and hence it will be assumed in what follows that transformation (3) is properly restricted.

Variables ρ and θ can be thought of as forming a curvilinear coordinate system for the z-plane, with $\rho = 1$ corresponding to the curvilinear contour Σ. In terms of ρ and θ, the displacements and stresses are

$$\left.\begin{aligned} 2\mu(u+iv) &= \kappa\phi(\zeta)-\omega(\zeta)\overline{\Phi(\zeta)}-\overline{\psi(\zeta)}, \\ \sigma_{\rho\rho}-i\sigma_{\rho\theta} &= \Phi(\zeta)+\overline{\Phi(\zeta)}-\frac{\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\{\overline{\omega(\zeta)}\Phi'(\zeta)+\omega'(\zeta)\Psi(\zeta)\}, \\ \sigma_{\rho\rho}+\sigma_{\theta\theta} &= 2[\Phi(\zeta)+\overline{\Phi(\zeta)}], \end{aligned}\right\} \tag{4}$$

where

$$\Phi(\zeta) = \frac{\phi'(\zeta)}{\omega'(\zeta)}, \qquad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}. \tag{5}$$

From (4), it is seen that the stresses will, in general, be singular at the points

$$|\omega'(\zeta)| = 0, \tag{6}$$

which are the singular points of transformation (3). Furthermore, (3) maps points inside $|\zeta| = 1$ all over the z-plane, that is, points inside the circle $\rho = 1$ do not necessarily correspond to points inside Σ. This precludes the use of the classical conformal mapping technique [1, 2] in solving elastic curvilinear inclusion problems.

The difficulties are readily avoided by modifying the formulation of the problem. Instead of attempting to find a solution by simultaneously satisfying the boundary conditions at infinity and the matrix–inclusion interface, the problem will be approached as follows: first, a representation of the solution that satisfies the boundary conditions at the interface, with no attention being paid to the conditions at infinity, is constructed. Next, the given problem is solved by selecting the appropriate combination of terms from the general solution. This approach has the advantage that the problem can be reduced to one of purely algebraic manipulations.

The general representation of the solution will now be found. This is most readily done by assuming the elastic fields inside the inclusion to be specified. This has the effect of fixing the values of the tractions and displacements at the interface and, hence, reducing the problem to one of finding the elastic fields exterior to a curvilinear region with prescribed surface tractions and displacements. This is a classical problem of the plane theory of elasticity, which is readily solved by well-known techniques. If the elastic fields inside the inclusion are taken to be *arbitrary*, the resulting solution is a general representation of the solution of the curvilinear inclusion problem. This approach is used below to generate some general results.

Inclusion problems

Consider an elastic matrix containing a single curvilinear inclusion whose boundary is given by Σ. Letting subscripts 1 and 2 on elastic quantities refer to the matrix and inclusion respectively, the boundary conditions corresponding to a perfect bond at the interface are

$$\left.\begin{aligned} u_1 + iv_1 &= u_2 + iv_2, \\ \sigma_{\rho\rho 1} - i\sigma_{\rho\theta 1} &= \sigma_{\rho\rho 2} - i\sigma_{\rho\theta 2}, \end{aligned}\right\} \quad \text{for } \rho = 1. \tag{7}$$

The complex potentials $\phi(\zeta)$ and $\psi(\zeta)$ satisfying (7) are

$$\phi_1(z) = \phi_1(\zeta) = f(z) - \frac{\beta-\alpha}{1-\beta}\frac{\omega(\zeta)}{\Omega'(\zeta)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\bar{f}[\Omega(\zeta)] - \frac{\beta-\alpha}{1+\beta}\bar{h}[\Omega(\zeta)],$$

$$\psi_1(z) = \psi_1(\zeta) = \frac{2\beta}{1-\beta}\frac{\Omega(\zeta)}{\omega'(\zeta)}\frac{\mathrm{d}}{\mathrm{d}\zeta}f[\omega(\zeta)] + \frac{\beta-\alpha}{1-\beta}\frac{\Omega(\zeta)}{\omega'(\zeta)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\omega(\zeta)}{\Omega'(\zeta)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\bar{f}[\Omega(\zeta)]\right]$$

$$+\frac{\alpha+\beta}{1-\beta}\bar{f}[\Omega(\zeta)] + h(z) + \frac{\beta-\alpha}{1+\beta}\frac{\Omega(\zeta)}{\omega'(\zeta)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\bar{h}[\Omega(\zeta)], \tag{8}$$

$$\phi_2(z) = \frac{1+\alpha}{1-\beta}f(z), \qquad \psi_2(z) = \frac{1+\alpha}{1+\beta}h(z),$$

where

$$\alpha = [\Gamma(\kappa_1+1)-(\kappa_2+1)]/[\Gamma(\kappa_1+1)+\kappa_2+1],$$
$$\beta = [\Gamma(\kappa_1-1)-(\kappa_2-1)]/[\Gamma(\kappa_1+1)+\kappa_2+1], \tag{9}$$
$$\Gamma = \mu_2/\mu_1$$

are the contractions suggested by Dundurs [18] and

$$\Omega(\zeta) = \omega(1/\zeta). \tag{10}$$

In deriving representation (8), the elastic fields inside the inclusion were assumed to be known. Equations (3) and (4) were used to write the displacements and stresses on Σ in terms of ρ and θ.† The boundary conditions were then satisfied at the interface with no attention being paid to the conditions at infinity.

It should be noted that general representation (8) of the solution depends on only two elastic parameters α and β and not on the four elastic constants κ_1, κ_2, μ_1 and μ_2. This is in agreement with the results recently reported by Dundurs [18].

Limiting cases

General solution (8) can be readily specialized for particular combinations of material properties. For a cavity or an inclusion of vanishing rigidity, $\Gamma = 0$ and (8) reduces to

$$\begin{aligned} \phi(\zeta) &= f(\zeta)-\bar{h}(1/\zeta), \\ \psi(\zeta) &= h(\zeta)-\bar{f}(1/\zeta)+\frac{\Omega(\zeta)}{\omega'(\zeta)}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\bar{h}(1/\zeta)-f'(\zeta)\right]. \end{aligned} \tag{11}$$

For a rigid curvilinear inclusion, $\Gamma = \infty$ and (8) reduces to

$$\begin{aligned} \phi(\zeta) &= f(\zeta)+\frac{1}{\kappa}\bar{h}(1/\zeta), \\ \psi(\zeta) &= h(\zeta)+\kappa\bar{f}(1/\zeta)-\frac{\Omega(\zeta)}{\omega'(\zeta)}\left[\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}\zeta}\bar{h}(1/\zeta)+f'(\zeta)\right]. \end{aligned} \tag{12}$$

Representations (11) and (12) are new results and they seem to be easier to use than the corresponding integral formulations.

ELLIPTIC INCLUSION PROBLEM

Now let us consider some applications of the general results. Let the inclusion boundary be given by (3), with $|\zeta| = 1$, and assume that a homogeneous state of stress exists within the inclusion. Then the appropriate loading functions are

$$f(z) = Az, \qquad h(z) = Bz, \tag{13}$$

† This can always be done since conformal mapping (3) is not applied to the inclusion region. It is only used to write the stresses and displacements, due to the assumed elastic fields inside the inclusion, in terms of ρ and θ on Σ. This may be seen from the following argument: an unbounded region, with properties of the inclusion, is loaded so that the elastic fields inside the region, bounded by Σ, are equal to the assumed ones. The stresses and displacements on boundary Σ of the inclusion region are expressed in terms of ρ and θ by using conformal mapping (3), which applies in this case.

where A and B are real and complex constants, respectively. Substitution of (13) into general representation (8) yields

$$\begin{aligned}\phi_1(z) &= \phi_1(\zeta) = \frac{1-2\beta+\alpha}{1-\beta}Az - \frac{\beta-\alpha}{1+\beta}\bar{B}\Omega(\zeta),\\ \psi_1(z) &= \psi_1(\zeta) = \frac{4A\beta}{1-\beta}\Omega(\zeta) + Bz + \bar{B}\frac{\beta-\alpha}{1+\beta}\frac{\Omega(\zeta)\Omega'(\zeta)}{\omega'(\zeta)},\end{aligned} \tag{14}$$

where

$$\Omega(\zeta) = c\left[\zeta^{-1} + \sum_{n=1}^{\infty}\lambda_n\zeta^n\right]. \tag{15}$$

Upon computing the stresses corresponding to (14), one sees that terms with $n > 1$ induce stresses that are not constant at infinity. Hence, $\lambda_n = 0\ (n > 1)$ for constant stress at infinity and, consequently, $\omega(\zeta) = c[\zeta + \lambda_1\zeta^{-1}]$, which is the mapping for an ellipse. Thus it has been shown that *the ellipse and its geometric limits is the only inclusion shape for which a constant stress at infinity induces a constant state of stress within the inclusion.* This result was established by Stippes [16] in a different manner.

Furthermore, (14) indicates the stress field that has to be applied to the matrix for the stresses to be uniform inside the inclusion. For example, if the inclusion is hypotrochoidal,

$$\omega(\zeta) = \zeta + \lambda\zeta^{-n}, \qquad 0 \le \lambda \le 1/n \tag{16}$$

and the solution corresponding to a constant hydrostatic stress inside the inclusion is

$$\begin{aligned}\phi_1(z) &= \phi_1(\zeta) = \frac{1-2\beta+\alpha}{1+\alpha}z, \qquad \phi_2(z) = z,\\ \psi_1(z) &= \psi_1(\zeta) = \frac{4\beta}{1+\alpha}\{\zeta^{-1}+\lambda\zeta^n\}, \qquad \psi_2(z) = 0.\end{aligned} \tag{17}$$

For a triangular inclusion ($n = 2$), the imposed stress must be a combined uniform loading and bending. Now let us consider the case of an elliptic inclusion in more detail.

Elliptic inclusion disturbing a uniform stress field

For the sake of convenience, let the boundary Σ of the inclusion be given by

$$z = \omega(\zeta) = \zeta + \lambda\zeta^{-1}, \qquad 0 \le \lambda \le 1, \qquad \lambda \text{ real}, \tag{18}$$

with $|\zeta| = 1$. From the results presented above, it follows that the problem of an elliptic inclusion disturbing a uniform applied stress field is readily solved by assuming loading functions (13) and adjusting the coefficients A and B. The solution is

$$\begin{aligned}\phi_1(z) = \phi_1(\zeta) &= \tfrac{1}{4}(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\{z+2DL(M-1)(1-\lambda^2)\lambda\zeta^{-1}\}\\ &\quad -\tfrac{1}{2}(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})(1-\lambda^2)DL\zeta^{-1}+\sigma_{xy}^{\infty}[(1-\lambda^2)D/(1-D\lambda^2)]i\zeta^{-1},\end{aligned} \tag{19}$$

$$\begin{aligned}\psi_1(z) = \psi_1(\zeta) &= \tfrac{1}{2}(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})(M-1)L\{[1-(1-D)\lambda^2]\zeta^{-1}+D\lambda[\zeta^{-1}-\lambda^3\zeta]/[\zeta^2-\lambda]\}\\ &\quad +\tfrac{1}{2}(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})\{\zeta+L[1+2D(M-1)]\lambda\zeta^{-1}-DL[\zeta^{-1}-\lambda^3\zeta]/[\zeta^2-\lambda]\}\\ &\quad +i\sigma_{xy}^{\infty}\{\zeta+\lambda\zeta^{-1}/(1-D\lambda^2)+D(\zeta^{-1}-\lambda^3\zeta)/[(1-D\lambda^2)(\zeta^2-\lambda)]\},\end{aligned} \tag{20}$$

$$\phi_2(z) = \tfrac{1}{4}(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})LM(1+D\lambda^2)z+\tfrac{1}{2}(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})DLM\lambda z, \tag{21}$$

$$\psi_2(z) = -\tfrac{1}{2}(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})L(1-D)(M-1)\lambda z+\tfrac{1}{2}(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})(1-D)Lz+i\sigma_{xy}^{\infty}(1-D)z/(1-D\lambda^2), \tag{22}$$

where σ_{xx}^{∞}, σ_{yy}^{∞} and σ_{xy}^{∞} are the stresses applied at infinity and

$$L = 1/[1-D\lambda^2+2DM\lambda^2],$$
$$M = (1+\alpha)/(1+\alpha-2\beta),$$
$$D = (\beta-\alpha)/(1+\beta).$$

Equations (19–22) constitute the complete solution of the elliptic inclusion problem for uniform stresses at infinity. Even though the problem was previously investigated by a number of authors [11–15], it is believed that the present results are more tractable in the elastic constants.

Elliptic inclusion disturbing an arbitrary stress field

Now let us consider the problem of an elliptic inclusion disturbing a general type of elastic field. If the elastic field inside the inclusion is known, the solution follows directly from representation (8). Unfortunately, this information is not available *a priori*. This fact, however, does not preclude using representation (8) in solving problems for arbitrary elastic fields imposed in the matrix.

If the imposed elastic fields are such that no singularities would be expected to exist within the inclusion, then the elastic fields inside the inclusion may be expanded in a power series. If the solution were known, the coefficients in the series would be determined. In the problem under consideration, the fields inside the inclusion are not known; but, it is expected that they can be represented by a power series. This indicates the following procedure for solving the problem.

Assume that the elastic fields within the inclusion may be represented by a power series, that is, assume

$$f(z) = \sum_{n=1}^{\infty} \{a_n z^n + ib_n z^n\}, \qquad h(z) = \sum_{n=1}^{\infty} \{c_n z^n + id_n z^n\}, \tag{23}$$

where a_n, b_n, c_n and d_n are real unknown coefficients. The stresses and displacements in the matrix, corresponding to (23), are found by using representation (8). Upon equating the fields, corresponding to (23), to the specified applied elastic fields in the matrix, a system of algebraic equations is found for the unknown coefficients. Thus the problem is solved in principle.

As an illustration of this procedure, consider the problem of an elliptic inclusion disturbing a bending field. For the sake of simplicity, let us assume that the bending field is given by

$$\sigma_{yy}^{\infty} = x, \qquad \sigma_{xy}^{\infty} = \sigma_{xx}^{\infty} = 0 \tag{24}$$

and the inclusion boundary is specified by (18). If the solid were homogeneous, the complex potentials corresponding to (24) would be

$$\phi^{\infty}(z) = \tfrac{1}{8}z^2, \qquad \psi^{\infty}(z) = \tfrac{1}{8}z^2. \tag{25}$$

Since the imposed potentials are functions of z with real coefficients, it is sufficient to take the loading functions as

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad h(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{26}$$

Upon using representation (8), it follows that

$$\phi_1(z) = \sum_{n=1}^{\infty} a_n\left\{z^n - n\frac{\beta-\alpha}{1-\beta}\zeta^{2-n}[\lambda\zeta^{-2} + \mathscr{L}_{nr}(\lambda^r\zeta^{2r})]\right\} - \frac{\beta-\alpha}{1+\beta}\sum_{n=1}^{\infty} c_n \sum_{r=0}^{n}\binom{n}{r}\zeta^{2r-n}\lambda^r , \tag{27}$$

$$\begin{aligned}\psi_1(z) = \sum_{n=1}^{\infty} a_n\Bigg\{&\frac{2n\beta}{1-\beta}\zeta^{n-2}[\lambda\zeta^2 + \mathscr{L}_{nr}(\lambda^r\zeta^{-2r})] + \frac{(n+1)\beta-(n-1)\alpha}{1-\beta}\sum_{r=0}^{n}\binom{n}{r}\zeta^{2r-n}\lambda^r\\
&+n(n-1)\frac{\beta-\alpha}{1-\beta}\Bigg[\lambda^2\zeta^{-n} + \lambda\zeta^{2-n}\mathscr{L}_{nr}(\lambda^r\zeta^{2r})\\
&+(\lambda^2-1)\frac{\zeta^{-n}}{\zeta^2-\lambda} + (\lambda^2-1)\zeta^{-n}\mathscr{L}_{nr}\left(\lambda^{2r}\sum_{t=1}^{r}\lambda^{-t}\zeta^{2t}\right)\\
&+(\lambda^2-1)\frac{\zeta^{2-n}}{\zeta^2-\lambda}\mathscr{L}_{nr}(\lambda^{2r})\Bigg]\Bigg\}\\
&+\sum_{n=1}^{\infty} c_n\Bigg\{z^n + n\frac{\beta-\alpha}{1+\beta}\Bigg[\lambda\sum_{r=0}^{n}\binom{n}{r}\lambda^r\zeta^{2r-n} + (1+\lambda^2)^n\frac{(\lambda^2-1)\zeta^{-n}}{\zeta^2-\lambda}\Bigg]\\
&+(\lambda^2-1)\sum_{r=1}^{n}\sum_{t=r}^{n}\binom{n}{t}\lambda^{2t}\lambda^r\zeta^{2r-n-2}\Bigg\},\end{aligned} \tag{28}$$

$$\phi_2(z) = \frac{1+\alpha}{1-\beta}\sum_{n=1}^{\infty} a_n z^n, \tag{29}$$

$$\psi_2(z) = \frac{1+\alpha}{1+\beta}\sum_{n=1}^{\infty} c_n z^n, \tag{30}$$

where $\mathscr{L}_{nr}$ is a summation operator defined by

$$\mathscr{L}_{nr}(\,.\,) = \sum_{r=0}^{n-1}\binom{n-1}{r}(\,.\,) + \lambda^2\sum_{r=0}^{n-2}\binom{n-1}{r+1}(\,.\,). \tag{31}$$

Upon examining (27–30), it is seen that it is sufficient to take

$$f(z) = a_2 z^2, \qquad h(z) = c_2 z^2. \tag{32}$$

In this case,

$$\phi_1(z) = a_2\left\{\left(1-2\lambda\frac{\beta-\alpha}{1-\beta}\right)\zeta^2 + \lambda^2\zeta^{-2} - \frac{2(\beta-\alpha)}{1-\beta}\lambda\zeta^{-2}\right\} - c_2\frac{\beta-\alpha}{1+\beta}\left\{\zeta^{-2}+\lambda^2\zeta^2\right\}, \tag{33a}$$

$$\begin{aligned}\psi_1(z) = a_2\Bigg\{&4\beta\lambda + (5\beta-3\alpha)\lambda^2]\frac{\zeta^2}{1-\beta} + \frac{4\beta\lambda}{1-\beta}\zeta^{-2} + \frac{3\beta}{1-\beta}\zeta^{-2} + \frac{2(\beta-\alpha)}{1-\beta}\Bigg[\lambda^2\zeta^{-2} + (\lambda^2-1)\\
&\times\left(\frac{\zeta^{-2}}{\zeta^2-\lambda} + \frac{1+2\lambda^2}{\zeta^2-\lambda}\right)\Bigg]\Bigg\} + c_2\Bigg\{\left[1+2\lambda^3\frac{\beta-\alpha}{1+\beta}\right]\zeta^2 + \lambda^2\zeta^{-2} + \frac{2(\beta-\alpha)}{1+\beta}\\
&\times\left[\lambda\zeta^{-2} + (\lambda^2-1)(2\lambda+\lambda^3)\zeta^{-2} + (\lambda^2-1)(1+\lambda^2)^2\frac{\zeta^{-2}}{\zeta^2-\lambda}\right]\Bigg\},\end{aligned} \tag{33b}$$

$$\phi_2(z) = \frac{1+\alpha}{1-\beta}a_2 z^2, \qquad \psi_2(z) = \frac{1+\alpha}{1+\beta}c_2 z^2, \tag{33c}$$

where terms that do not contribute to the stresses and displacements have been dropped. Equating the coefficients of ζ^2 in (33a, b) and (25), one gets the following algebraic equations for the two unknown coefficients:

$$[1-2\lambda(\beta-\alpha)/(1-\beta)]a_2-\lambda^2[(\beta-\alpha)/(1+\beta)]c_2 = \tfrac{1}{8}$$
$$[4\beta\lambda+(5\beta-3\alpha)\lambda^2]a_2/(1-\beta)+[1+2\lambda^3(\beta-\alpha)/(1+\beta)]c_2 = \tfrac{1}{8}.$$

Solving these equations completes the solution of the problem. Other cases of assumed applied stresses may be handled just as easily; but the computations might get rather messy.

CONCLUSION

The method used to discuss the elliptic inclusion problem may be used to solve problems for other inclusion shapes. By using representation (8), a sequence of solutions corresponding to loading functions $f(z)$ and $h(z)$, given by (23) within the inclusion, can be generated. In contrast to the case of an elliptic inclusion, the stresses in the matrix will contain terms with higher powers of ζ than those in the assumed field; e.g. $f(z) = z$ will give stresses in the matrix that contain higher order terms in addition to the constant term. This leads to the situation in which one has to use the complete power series expansion for $f(z)$ and $h(z)$, unless the imposed fields are unrealistically contrived. Thus for a given imposed field use of the proposed method will lead to an infinite system of algebraic equations that can be solved only approximately by truncation techniques. Hence, it seems that only approximate solutions can be found for the general curvilinear inclusion problem. This is exactly the situation that is encountered in curvilinear inclusion problems in the case of antiplane deformation [19].

The use of the proposed analysis method is not restricted to an unbounded matrix. If the matrix is finite, representation (8) may be used to generate a sequence of stress functions satisfying the boundary conditions at the matrix–inclusion interface. The conditions at the other boundaries may then be satisfied by a point matching, boundary collocation, or some other method for satisfying boundary conditions approximately. Thus the results presented above may be used in principle to get improved solutions for finite solids containing inclusions.

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(*Received* 7 *August* 1969; *revised* 24 *November* 1969)

Абстракт—Дается общее представление решения для упругого криволинейного включения. Показано, что эллипс и его геометрические границы являются единственной формой включения, для которой равномерное напряжение, приложенное в бесконечности, вызывает постоянное напряженное состояние внутри включения. Иллюстрируется применение общего решения некоторым числом примеров.